Linear 1st Order Differential Equations

Kutay Guven

April 18, 2024

NB the integrating factor, where $\frac{dy}{dx} + p(x)y = q(x)$, is defined as $I(x) = \exp \circ P$ where $\circ$ denotes composition and $P(x)$ is the indefinite integral of $p(x)$ with respect to x. Furthermore, it can be shown that any linear 1st order ordinary differential equation has a general solution given by

$$y = \frac{\int q(x)dx}{I(x)}$$

1. Find the general solution (for y) of each system where $y' := \frac{dy}{dx}$.
   (a) $y' - y = 0$
   (b) $y' + xy = 0$
   (c) $y' - x^{49}y = 15$
   (d) i. $10y' - \frac{150y}{x} = 100x^2$
       ii. $10y' - \frac{150y}{x} = 100y^2$ (Hint: This is a Bernoulli differential equation.)

2. A certain fitness club has a yearly profit function in gems given by $\frac{dP}{dt} - \frac{P}{2} = t^{0.2}$ with respect to time in years since it opened back in the year 1949 AD. You are given that the fitness club made 0 gems of profit in the year it was founded, and gems are a universal currency that has always and will always have the same value.
   (a) Find the general solution for P in gems, and hence the particular solution for P.
   (b) If, instead of 1949 AD, the fitness club was founded back in 1949 BC, find the exact year where P would first surpass 10^{12} kilogems.
   (c) Explain, using your solution to (b) or otherwise, why this profit function may be unreasonable.

3. A particle moves along a line according to the equation

   $$(t+1)v^2a + v^3 = (t+1)v^4 \ln(5t+1)$$

   where a, v, t are its acceleration and deceleration and time elapsed (in seconds) from the start of the movement of the particle respectively.
   (a) Given that the particle's initial velocity is $\frac{5}{76}ms^{-1}$, find the particular solution for v.
   (b) State $\lim_{t\to\infty} \frac{1}{v}$ and hence comment on whether this model is reasonable.